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Quantization of systems with a general phase space equipped with a Riemannian metric

Robert Alicki†§ and John R Klauder‡‖

† Institute of Theoretical Physics and Astrophysics, University of Gdańsk, PL-80-952 Gdańsk, Poland

‡ Departments of Physics and Mathematics, University of Florida, Gainesville, FL 32611, USA

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Abstract. Quantization on phase spaces of general geometry devoid of any special symmetry properties is discussed on the basis of phase spaces endowed with a symplectic structure, a Riemannian geometry, and a $Spin^c$ structure. Using techniques from differential geometry, and especially exploiting the Dirac operator, we are able to offer a fully geometric quantization procedure for a wide class of symmetry free phase spaces. Our procedure leads to the conventional results in cases where the phase space is a symmetric space for which alternative quantization techniques suffice.

1. Introduction

Quantization, as originally conceived by Schrödinger for example, was limited in its applicability to Euclidean phase spaces. The introduction of kinematical groups other than the Heisenberg–Weyl group has been the key to extending quantization to a variety of symmetric-space phase spaces, be it by coherent-state methods [1], Berezin quantization [2], deformation quantization [3], techniques of Isham [4], or other closely related techniques. Symmetric spaces possess a high degree of symmetry, and efforts to quantize systems on phase spaces with more general geometry having little or no symmetry have been introduced only recently [5–8]. All of the procedures mentioned rely, in one way or another, on adding a Riemannian metric to the symplectic phase space of classical systems. Our previous work [6] in this direction dealt with rather general spaces, but was confined to a two-dimensional phase space, i.e. a single degree of freedom. In the present paper we add a further structure to the space—namely a $Spin^c$ structure [9]—and show, for multidimensional phase spaces that admit the required structures, how the process of quantization may be extended to cases in which the phase space exhibits no symmetry whatsoever.

2. Geometry of classical systems

In this paper we restrict ourselves to a classical system with a phase space M being a $2n$-dimensional manifold (without a boundary). The kinematics is given by a non-degenerate and closed two-form (symplectic form), Ω; $d\Omega = 0$, which is globally defined on M while

§ E-mail: fizra@halina.univ.gda.pl

‖ E-mail: klauder@phys.ufl.edu

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the dynamics is given in terms of a Hamiltonian function $h : M \to R$. The Hamiltonian equations of motion can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Omega^{-1}\,\mathrm{d}h \tag{2.1}$$

or in local coordinates

$$\Omega_{\mu\nu}\dot{x}^{\nu} = \partial_{\mu}h. \tag{2.2}$$

The Darboux theorem states that for any $x \in M$ there exists a local coordinate system $(q_j, p_j; j = 1, \ldots, n)$ such that $\Omega = \sum \mathrm{d}q_j \wedge \mathrm{d}p_j$. Therefore in this coordinate system equations (2.1) and (2.2) take the standard form

$$\dot{q}_j = \frac{\partial h}{\partial p_j} \qquad \dot{p}_j = -\frac{\partial h}{\partial q_j} \qquad j = 1, 2, \ldots, n. \tag{2.3}$$

There exists (locally) a one-form $\Theta = \Theta_{\mu}\,\mathrm{d}x^{\mu}$ such that $\Omega = \mathrm{d}\Theta$ and which is defined up to a gauge transformation $\Theta \mapsto \Theta + \mathrm{d}f$. The Hamiltonian equations of motion are obtained from the variational principle for the action

$$\mathcal{S} = \int_{\gamma}[\Theta - h\,\mathrm{d}t] \qquad \gamma : [t_1, t_2] \to M \tag{2.4}$$

or in local coordinates

$$\mathcal{S} = \int_{t_1}^{t_2}[\Theta_{\mu}(x(t))\dot{x}^{\mu}(t) - h(x(t))]\,\mathrm{d}t. \tag{2.5}$$

3. Geometry of quantized systems

The quantization procedure proposed in the next section involves additional structure superimposed on the symplectic manifold (M, Ω). This is the so-called *generalized spin structure* or *Spin*^{c} *structure* (see [9] p 369 and references therein) which combines in a generally non-trivial way a (local) spin stucture over a Riemannian manifold and a $U(1)$ principal bundle structure.

Riemannian structure. We assume that there exists a Riemannian metric $(\cdot, \cdot)$ on the tangent bundle TM. The associated *orthonormal frame bundle* is denoted by $O(M) = \{r = (x, e_1, \ldots, e_{2n}) | e_j$ *form an orthonormal frame of* $T_x M\}$. The Riemannian structure defines a unique Levi-Civita connection ∇ on M which can be lifted to any bundle associated with $O(M)$.

Spin structure on Riemannian manifold. If M is an orientable Riemannian manifold, the *Clifford bundle* $C(M)$ is the bundle over M whose fibre at $x \in M$ is the Clifford algebra $C(T_x^* M)$ generated by the elements $c(v)$, $v \in T_x^* M$. Here $c(\cdot)$ is a linear map satisfying the anticommutation relations

$$c(v)c(u) + c(u)c(v) = -2(u, v). \tag{3.1}$$

The spinor representation of the Clifford algebra is constructed in the following way. Let Ξ denote an n-dimensional complex Hilbert space with the orthonormal basis $\{\xi_s = e_{2s-1} - \mathrm{i}e_{2s} | 1 \leqslant s \leqslant n\}$. The spinor space S can be seen as a fermionic Fock space over the Hilbert space Ξ generated by the orthonormal basis $\{\xi_{s_1} \wedge \ldots \wedge \xi_{s_p}; 0 \leqslant p \leqslant n\}$ of 'p-particle vectors'. The complexification of the Clifford algebra $C(T_x^* M)$ is isomorphic to the algebra $\mathcal{B}(S)$ of linear operators on S by the following identification,

$$c(e^{2s-1}) = a_s + a_s^* \qquad c(e^{2s}) = \mathrm{i}(a_s^* - a_s) \tag{3.2}$$

where a_s, a_s^* are fermionic annihilation and creation operators defined in terms of the orthonormal basis $\{\xi_s\}$, and $(e^1, \ldots, e^{2n})$ is an orthonormal frame of T_x^*M which can be identified with $(e_1, \ldots, e_{2n})$.

The spinor bundle inherits the unique Levi-Civita spinor connection ∇^S which in a local orthonormal frame is given by

$$\nabla_\mu^S = \partial_\mu + \tfrac{1}{4}\omega_{\mu lk} c(e^l) c(e^k) \tag{3.3}$$

where $\nabla_\mu e_k = \omega_{\mu}{}^{l}{}_{k} e_l$, and $\omega_{\mu lk} = \omega_{\mu}{}^{l}{}_{k}$.

Generalized spinor structure: $Spin^c$ structure. The construction presented above defines a local spin bundle together with the local action of the Clifford bundle. In order to perform our quantization procedure (see the next section) we need a global $Spin^c$ bundle over M which will be denoted by $S^c(M)$. There are topological obstructions to the existence of a $Spin^c$ structure, namely the second Stiefel–Whitney class must be the mod_2 reduction of an integer class.

4. The quantization procedure

We propose the following quantization procedure for a classical system with a phase space (M, Ω).

Definition 1. A classical system with a phase space (M, Ω) is *quantizable* if there exists a $Spin^c$ structure over M and the covariant derivative on the $Spin^c$ bundle $S^c(M)$ exists locally in the form $\nabla^S + (\mathrm{i}/\hbar)\Theta$ such that $\mathrm{d}\Theta = \Omega$. Any such structure defines a particular quantization of (M, Ω).

For quantizable systems we now construct the basic ingredients of the quantum theory: the Hilbert space $\mathcal{H}$ and the quantization map for observables.

Let $L^2(S^c(M))$ denote the Hilbert space of square integrable sections of the $Spin^c$ bundle $S^c(M)$. In local coordinates one can view the elements of $L^2(S^c(M))$ as spinor-valued functions with the scalar product

$$\langle\Psi|\Phi\rangle = \int \Psi^\dagger(x)\Phi(x)\sqrt{g}\,\mathrm{d}x \qquad \Psi, \Phi \in L^2(S^c(M)). \tag{4.1}$$

Definition 2. The *Hilbert space* $\mathcal{H}$ of the quantized system (M, Ω) corresponding to the $Spin^c$ bundle $S^c(M)$ is defined as the kernel of the geometric Dirac operator acting on $L^2(S^c(M))$

$$\mathcal{D}_\Theta = c(\mathrm{d}x^\mu)[\nabla_\mu^S + (\mathrm{i}/\hbar)\Theta_\mu] \tag{4.2}$$

i.e.

$$\mathcal{H} = \ker(\mathcal{D}_\Theta) = \{\Psi; \Psi \in L^2(S^c(M)), \mathcal{D}_\Theta\Psi = 0\}. \tag{4.3}$$

Definition 3. A *quantum observable* $\hat{f}$ corresponding to a classical one $f : M \to \mathbb{R}$ is given by the following *quantization map*,

$$f \to \hat{f} = \Pi f \Pi \tag{4.4}$$

where $\Pi : L^2(S^c(M)) \to \mathcal{H}$ is an orthogonal projection and f on the right-hand side of equation (4.4) is treated as a multiplication operator on $L^2(S^c(M))$.

The proposed quantization scheme generalizes and unifies two ideas: the polarization formula in the geometric quantization approach [10, 11] and the Toeplitz quantization map commonly employed in coherent state techniques [1, 12], and recently used in the context of non-commutative geometry [13]. Moreover, in section 6 we will construct a regularized path integral formula for the unitary evolution operator $\exp(-(\mathrm{i}/\hbar)\hat{h}t)$ extending the ideas developed in [1, 6, 12, 14].

5. Atiyah–Singer index theorem and the non-triviality of quantization

The natural question arises whether the proposed quantization scheme leads to a non-trivial Hilbert space $\mathcal{H}$ with an 'appropriate' dimension. For example, for a compact phase space the dimension of $\mathcal{H}$ should be roughly proportional to the 'volume' of M. For non-compact M the density of quantum states $\mathrm{d}N$ makes sense, and for the standard case of $M = \mathbb{R}^{2n}$ with $\Omega = \sum \mathrm{d}q_j \wedge \mathrm{d}p_j$ we have

$$\mathrm{d}N = \prod_{k=1}^{n} \frac{\mathrm{d}q_k\, \mathrm{d}p_k}{2\pi\hbar} = [(2\pi\hbar)^n n!]^{-1} \bigwedge_n \Omega. \tag{5.1}$$

In the case of a general but compact phase space manifold M we can estimate the dimension of $\mathcal{H}$ using the celebrated Atiyah–Singer index theorem [15] . The index of the Dirac operator $\mathcal{D}_\Theta$ is defined as

$$\mathrm{index}(\mathcal{D}_\Theta) = \dim(\ker \mathcal{D}_\Theta^+) - \dim(\ker \mathcal{D}_\Theta^-) \tag{5.2}$$

where

$$\mathcal{D}_\Theta = \begin{pmatrix} 0 & \mathcal{D}_\Theta^+ \\ \mathcal{D}_\Theta^- & 0 \end{pmatrix}$$

and this grading of the Dirac operator comes from the natural grading of the spinor space $S = S^+ \bigoplus S^-$. Obviously

$$\dim \mathcal{H} \geqslant |\,\mathrm{index}(\mathcal{D}_\Theta)| \tag{5.3}$$

and the Atiyah–Singer theorem gives

$$\mathrm{index}(\mathcal{D}_\Theta) = \int_M [\mathrm{ch}(\Theta/\hbar) \cdot \hat{A}(M)]_{\mathrm{top}} \tag{5.4}$$

where the so-called $\hat{A}$ *genus*

$$\hat{A}(M) = \prod_{k=1}^{n} \left[\frac{R_k/4\pi}{\sinh(R_k/4\pi)} \right] \tag{5.5}$$

is a differential form defined in terms of 2-forms R_k which are obtained by block diagonalizing the curvature

$$R = \tfrac{1}{2} R_{kl\gamma\lambda}\, \mathrm{d}x^\gamma \wedge \mathrm{d}x^\lambda = \mathrm{diag} \begin{pmatrix} 0 & R_k \\ -R_k & 0 \end{pmatrix}.$$

The Chern character of $\Theta/\hbar$ is given by

$$\mathrm{ch}(\Theta/\hbar) = \exp \frac{\Omega}{2\pi\hbar}. \tag{5.6}$$

In equation (5.4) 'top' means that the highest rank $(2n)$-form in the power series expansion is integrated. In the semiclassical limit $\hbar \to 0$ the leading term is given by

$$\mathrm{index}(\mathcal{D}_\Theta) \approx \frac{1}{(2\pi\hbar)^n n!} \int_M \bigwedge_n \Omega \tag{5.7}$$

which clearly corresponds to equation (5.1) and assures the non-triviality of the quantization. For non-compact manifolds the local index theorems [15] still make sense and one can obtain the following estimate for the density of quantum states:

$$\mathrm{d}N(x) \geqslant |\rho(x)|\,\mathrm{d}x \tag{5.8}$$

where $\rho(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n = [\mathrm{ch}(\Theta/\hbar) \cdot \hat{A}(M)]_{\mathrm{top}}$.

6. Regularized path integral

In [12] the idea of stochastically regularized coherent-state path integrals has been carefully studied for the case of $M = \mathbb{R}^{2n}$. Thereafter the formalism was generalized to examples where (i) M is a homogeneous manifold for certain Lie groups [2], (ii) M is a Kähler manifold [14], and (iii) M is a general two-dimensional Riemannian surface [6]. In the following we modify and extend these ideas to our most general situation.

We begin with the construction of diffusions on Riemannian manifolds and their horizontal lifts [16].

Let $W(t) = (W^j(t);\, j = 1, 2, \ldots, 2n)$ be a canonical normalized $2n$-dimensional Wiener process with expectation $\mathcal{E}$. We consider a stochastic differential equation in the Stratonovitch sense on the frame bundle $O(M)$ which can be written in local coordinates as

$$\begin{aligned}
\mathrm{d}x^\mu(t) &= \sqrt{\kappa}\, e^\mu_k(t) \circ \mathrm{d}W^k(t) \\
\mathrm{d}e^\mu_k(t) &= -\Gamma^\mu_{\lambda\nu}(x(t)) e^\nu_k(t) \circ \mathrm{d}x^\lambda(t)
\end{aligned} \tag{6.1}$$

with the initial conditions $x(0) = x$, $e_k(0) = e_k$. Here $\Gamma^\mu_{\lambda\nu}$ are Christoffel symbols for the Levi-Civita connection and $\kappa > 0$ is a diffusion constant. The solution of equations (6.1) exists and defines a stochastic diffusion process $r^{(\kappa)}(t) = (x^\mu(t), e^\mu_k(t), \mu; k = 1, 2, \ldots, 2n)$ on $O(M)$. We shall use a notation which explicitly shows the κ-dependence, i.e. $\{x^\mu(t)\} = x(t) \equiv x^{(\kappa)}(t)$ etc.

One can easily check that the stochastic process $x(t)$ alone is the canonical diffusion process on a base manifold M governed by the following diffusion equation for $f(x;t) = \mathcal{E}(f(x^{(\kappa)}(t))$:

$$\frac{\partial}{\partial t} f(x;t) = \frac{\kappa}{2} \Delta f(x;t) \tag{6.2}$$

where Δ denotes the Beltrami–Laplace operator.

There exists a canonical way of lifting the diffusion process (6.1) to fibre bundles associated with $O(M)$ such as a tensor bundle, a differential form bundle or a spinor bundle. The last example is relevant for us. Let $\Psi(x)$ be a spinor field on M and by $\Psi[r]$ we denote its representation in terms of the orthonormal frame $r = (x, e_k)$. Namely, we can identify $\Psi[r]$ with the coefficients of $\Psi(x)$ in the natural orthonormal basis in S, i.e.

$$\Psi[r] \equiv \{< \xi_{w_1} \wedge \ldots \wedge \xi_{w_p}, \Psi(x) >;\, 0 \leqslant p \leqslant n\}. \tag{6.3}$$

Then defining $\Psi(x;t)$ in a local reference frame by

$$\Psi[r;t] = \mathcal{E}\Psi[r^{(\kappa)}(t)] \tag{6.4}$$

one may check that $\Psi(x;t)$ satisfies the following diffusion equation:

$$\frac{\partial}{\partial t}\Psi(x;t) = \frac{\kappa}{2}\Delta^{(\mathrm{B})}\Psi(x;t). \tag{6.5}$$

Here $\Delta^{(\mathrm{B})}$ denotes the *horizontal Bochner Laplacean* for spinors given in local coordinates by

$$\Delta^{(\mathrm{B})} = g^{\mu\nu}(x)(\nabla^S_\mu \nabla^S_\nu - \Gamma^\lambda_{\mu\nu}(x)\nabla^S_\lambda). \tag{6.6}$$

In order to derive a path integral expression for the quantum propagator $\exp(-(\mathrm{i}/\hbar)\hat{h}t)$ on $\mathcal{H}$ we use first a quite general formula [12, 17] which in our case reads

$$\exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{h}t\right\} = \Pi \exp\left\{-\frac{\mathrm{i}}{\hbar}\Pi h \Pi t\right\}\Pi = s - \lim_{\kappa\to\infty}\exp\left\{\left[\frac{\kappa}{2}\mathcal{D}^2_\Theta - \frac{\mathrm{i}}{\hbar}h\right]t\right\}. \tag{6.7}$$

The next step is to apply the Lichnerowicz theorem [15] which gives the following decomposition

$$\mathcal{D}^2_\Theta = \Delta^{(B)}_\Theta + \frac{r_M}{4} + \frac{1}{2\hbar}\boldsymbol{c}(\Omega) \tag{6.8}$$

where $\Delta^{(\mathrm{B})}_\Theta$ is a twisted Bochner Laplacean obtained from (6.6) by replacing ∇^S with $\nabla^S + (\mathrm{i}/\hbar)\Theta$, r_M is a scalar curvature of M, and in a coordinate system

$$\boldsymbol{c}(\Omega)[r] = \frac{\mathrm{i}}{2}\Omega_{kl}(r)c(e_k)c(e_l) \qquad \Omega_{kl}(r) = \Omega_{\mu\nu}(x)e^\mu_k(x)e^\nu_l(x). \tag{6.9}$$

Now using equations (6.5)–(6.9) we can apply the Feynman–Kac formula to get the following regularized path integral expression:

$$\begin{aligned}(\mathrm{e}^{-(\mathrm{i}/\hbar)\hat{h}t}\Psi)[r] = \lim_{\kappa\to\infty}\mathcal{E}\Bigg\{&\exp\left[\frac{\mathrm{i}}{\hbar}\int(\Theta_\mu\,\mathrm{d}x^\mu - h\,\mathrm{d}t)\right]\\ &\times\boldsymbol{T}\exp\left[\frac{\kappa}{4}\int_0^t\left(\frac{1}{2}r_M(x^{(\kappa)}(s)) + \frac{1}{\hbar}\boldsymbol{c}(\Omega)[r^{(\kappa)}(s)]\right)\mathrm{d}s\right]\Psi[r^{(\kappa)}(t)]\Bigg\}\end{aligned} \tag{6.10}$$

with $\boldsymbol{T}$ being the time ordering operator and $\Psi \in \mathcal{H}$.

In formula (6.10) one can identify three basic elements:

(1) the Feynman probability amplitude $\mathrm{e}^{(\mathrm{i}/\hbar)\mathcal{S}}$;

(2) the geometric κ-dependent 'corrections' which involve both curvatures r_M and Ω;

(3) the regularizing diffusion process on $O(M)$ which gives a mathematically rigorous meaning to the path integral for all $0 < \kappa < \infty$.

In the general case formula (6.10) is rather complicated. In the next section we discuss examples for which the structure of $\mathcal{H}$ is much more explicit and the path integral (6.10) can be dramatically simplified.

However, even in the general case one can replace the time-ordered matrix-valued term in (6.10) by another regularized path integral with respect to auxiliary variables. Namely, we have the following identities for the relevant matrix elements between two arbitrary spinors χ', χ'' treated as elements of $\boldsymbol{C}^{2^n}$

$$\begin{aligned}&\chi''^\dagger\boldsymbol{T}\exp\left[\frac{\kappa}{4}\int_0^t\left(\frac{1}{2}r_M(x^{(\kappa)}(s)) + \frac{1}{\hbar}\boldsymbol{c}(\Omega)[r^{(\kappa)}(s)]\right)\mathrm{d}s\right]\chi'\\ &\qquad = \lim_{a\to0 b\to0}(ab)^{-1}[\langle a\chi'', t|b\chi', 0\rangle - 1]\end{aligned} \tag{6.11}$$

where

$$\begin{aligned}\langle\chi'', t|\chi', 0\rangle \equiv \exp\Bigg\{&-\frac{1}{2}\chi''^\dagger\chi'' - \frac{1}{2}\chi'^\dagger\chi'\\ &+\chi''^\dagger\boldsymbol{T}\exp\left[\frac{\kappa}{4}\int_0^t\left(\frac{1}{2}r_M(x^{(\kappa)}(s)) + \frac{1}{\hbar}\boldsymbol{c}(\Omega)[r^{(\kappa)}(s)]\right)\mathrm{d}s\right]\chi'\Bigg\}\end{aligned}$$

$$= \lim_{\nu\to\infty} \mathrm{e}^{2^n \nu t/2} \int \exp\left\{ \frac{1}{2}\int_0^t [\chi^\dagger(s)\,\mathrm{d}\chi(s) - (\mathrm{d}\chi^\dagger(s))\chi(s)] \right.$$
$$\left. + \frac{\kappa}{4}\int_0^t \left[\frac{1}{2} r_M(x^{(\kappa)}(s))\chi^\dagger(s)\chi(s) + \frac{1}{\hbar}\chi^\dagger(s)\boldsymbol{c}(\Omega)[r^{(\kappa)}(s)]\chi(s)\right]\mathrm{d}s\right\} \mathrm{d}\mathcal{P}_W^\nu(\chi) \tag{6.12}$$

and $\mathcal{P}_W^\nu$ denotes Wiener measure on $\boldsymbol{C}^{2^n}$ concentrated on continuous paths $\chi(s)$, $0 \leqslant s \leqslant t$, pinned so that $\chi(0) = \chi'$ and $\chi(t) = \chi''$, and with a transition probability given by

$$\int \mathrm{d}\mathcal{P}_W^\nu(\chi) = (2\pi\nu t)^{-2^n} \exp\left(-\frac{|\chi'' - \chi'|^2}{2\nu t}\right) \tag{6.13}$$

an expression that shows ν to be the diffusion constant. We observe in addition that the expression

$$\langle\chi''|\chi'\rangle \equiv \exp[-\tfrac{1}{2}\chi''^\dagger\chi'' + \chi''^\dagger\chi' - \tfrac{1}{2}\chi'^\dagger\chi'] \tag{6.14}$$

which is just (6.12) for $t \equiv 0$, is a positive definite function, and according to the GNS theorem, and as suggested by the notation, this expression may be interpreted as the inner product of two vectors of the form

$$|\chi\rangle = \prod_{l=1}^{2^n} |\chi_l\rangle \qquad |\chi_l\rangle = \mathrm{e}^{-\frac{1}{2}\chi_l^*\chi_l} \sum_{n_l=0}^{\infty} (n_l!)^{-1/2}\chi_l^{n_l}|n_l\rangle$$
$$\langle n_l|n'_{l'}\rangle = \delta_{ll'}\delta_{nn'}. \tag{6.15}$$

These are just the usual canonical coherent states, which admit a resolution of unity in the form

$$\mathbf{1} = \int |\chi\rangle\langle\chi| \prod_{l=1}^{2^n} \mathrm{d}\,\mathrm{Re}\,\chi_l\,\mathrm{d}\,\mathrm{Im}\,\chi_l/\pi. \tag{6.16}$$

Combining now formulae (6.10) with (6.11) and (6.12) we obtain a fully 'scalarized' regularized path integral expression for the quantum propagator.

7. Examples

In this section we briefly discuss two classes of phase spaces which have been treated previously [6, 14] using an approach based on the prequantization Hilbert space which consists of square integrable sections of the *line bundle* over M instead of the $Spin^c$ bundle used in the present paper. The results are slightly different but the difference vanishes in the semiclassical limit.

Two-dimensional phase space (compare with [6]). Locally one can always find a coordinate system $x^1 = u$, $x^2 = v$ such that the metric equals to

$$\mathrm{d}s^2 = e^{2w(u,v)}(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{7.1}$$

The natural choice of the orthonormal frame is

$$e^1 = e^w\,\mathrm{d}u \qquad e^2 = e^w\,\mathrm{d}v \tag{7.2}$$

and the spinor space S is spanned by the 'spin down' vector $(e_1 + \mathrm{i}e_2)/\sqrt{2} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the 'spin up' vector $(e_1 - \mathrm{i}e_2)/\sqrt{2} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence $c(e^1) = \sigma_1$ and $c(e^2) = \sigma_2$ where σ_k,

$k = 1, 2, 3$ are Pauli matrices. The spin connection $\omega^a_b = \omega_{\mu}{}^a_b \, \mathrm{d}x^\mu$ can be easily calculated using Cartan's relations: $\omega^a_b = -\omega^b_a$, $\mathrm{d}e^a = -\omega^a_b \wedge e^b$. One obtains

$$\omega_u{}^1_2 = -\omega_u{}^2_1 = \frac{\partial w}{\partial v} \qquad \omega_v{}^1_2 = -\omega_v{}^2_1 = -\frac{\partial w}{\partial u}. \tag{7.3}$$

A straightforward calculation yields

$$\mathcal{D}^{\pm}_{\Theta} = e^{-w}\left[(\partial_u \pm \mathrm{i}\partial_v) - \frac{\mathrm{i}}{\hbar}(\Theta'_u \pm \mathrm{i}\Theta'_v\right] \tag{7.4}$$

with

$$\Theta'_u = \Theta_u - \frac{\hbar}{2}\frac{\partial w}{\partial v} \qquad \Theta'_v = \Theta_v + \frac{\hbar}{2}\frac{\partial w}{\partial u}. \tag{7.5}$$

Now the analysis is exactly the same as in the [6]. Depending on the sign of the 'flux' $\int_M \Omega$, the Hilbert space $\mathcal{H}$ is spanned either by 'spin down' or 'spin up' functions $\psi_\pm$, and hence the inequality (5.8) becomes an equality. The only difference is that in all formulae Θ is replaced by Θ' given by equation (7.5). It leads to the cancellation of the r_M-dependent term in the density of quantum states which now reads

$$\mathrm{d}N = \pm\frac{1}{2\pi\hbar}\Omega. \tag{7.6}$$

In contrast to the quantizations based on the line bundle we obtain an *exact* correspondence between the symplectic volume of the phase space and the dimension of the Hilbert space. In particular, a classical spin with a phase space S^2 with volume $\int \Omega = 2\pi\hbar N$ is quantized to the spin-j representation such that $2j+1 = N$ while 'line bundle methods' give $2j = N$.

Kähler manifolds (compare with [14]). We assume that the phase space (M, Ω) is now a Kähler manifold such that if $z = (z^1, z^2, \dots, z^n)$ is a local chart of complex coordinates and $F(z, \bar{z})$ the (local) Kähler potential we have

$$\Theta = \mathrm{Im}\{\partial F\} \qquad \Omega = \mathrm{d}\Theta = \mathrm{i}\partial\bar{\partial}F \tag{7.7}$$

along with the Kähler metric

$$g_{k\bar{l}} = \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l}. \tag{7.8}$$

The complexification of the tangent bundle splits into two pieces, called the holomorphic $(T^{1,0}M)$ and antiholomorphic $(T^{0,1}M)$ ones. It leads to an invariant decomposition into annihilation and creation operators $a_{\bar{l}}$, a^*_k associated with local orthonormal frames. They annihilate a 'vacuum' ($|0\rangle$) or an 'antivacuum' ($|1\rangle$) vector in the spinor space S, respectively.

The structure of the Dirac operator is now the following,

$$\mathcal{D}_\Theta = \tfrac{1}{2}\{c(\mathrm{d}z^k)\bar{\nabla}^F_k + c(\mathrm{d}\bar{z}^k)\nabla^F_k\} \tag{7.9}$$

with

$$\bar{\nabla}^F_k = \frac{\partial}{\partial \bar{z}^k} + \frac{1}{2\hbar}\frac{\partial F}{\partial \bar{z}^k} + \frac{1}{4}\omega_{\bar{k}}{}^{\bar{r}}_s a_r a^*_s. \tag{7.10}$$

We have also $c(\mathrm{d}z^k)|1\rangle = 0$, $c(\mathrm{d}\bar{z}^k)|0\rangle = 0$. Hence one expects that generically the solutions of $\mathcal{D}_\Theta\Psi = 0$ are of the form $\Psi(x) = \psi(x)|0\rangle$ with a complex function ψ satisfying

$$\left(\frac{\partial}{\partial \bar{z}^k} + \frac{1}{2\hbar}\frac{\partial F}{\partial \bar{z}^k} + \frac{1}{4}\frac{\partial}{\partial \bar{z}^k}\ln g\right)\psi(z, \bar{z}) = 0 \tag{7.11}$$

and $g = g(z, \bar{z}) = \det[g_{\mu\nu}]$. The solutions of (7.11) are (locally) of the form (compare with [14])

$$\psi(z, \bar{z}) = \phi(z)\mathrm{e}^{-(1/2\hbar)F(z,\bar{z})}[g(z, \bar{z})]^{-\frac{1}{4}}. \tag{7.12}$$

The global extensions of the solutions (7.12) must satisfy square integrability conditions and topological constraints.

For both of the given examples of phase spaces, the Hilbert space $\mathcal{H}$ effectively consists of scalar complex valued functions and therefore the path integral (6.10) can be 'scalarized' also without the need to introduce auxiliary variables χ and interpreted directly in terms of generalized coherent states as in [1, 6, 12, 14, 18].

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